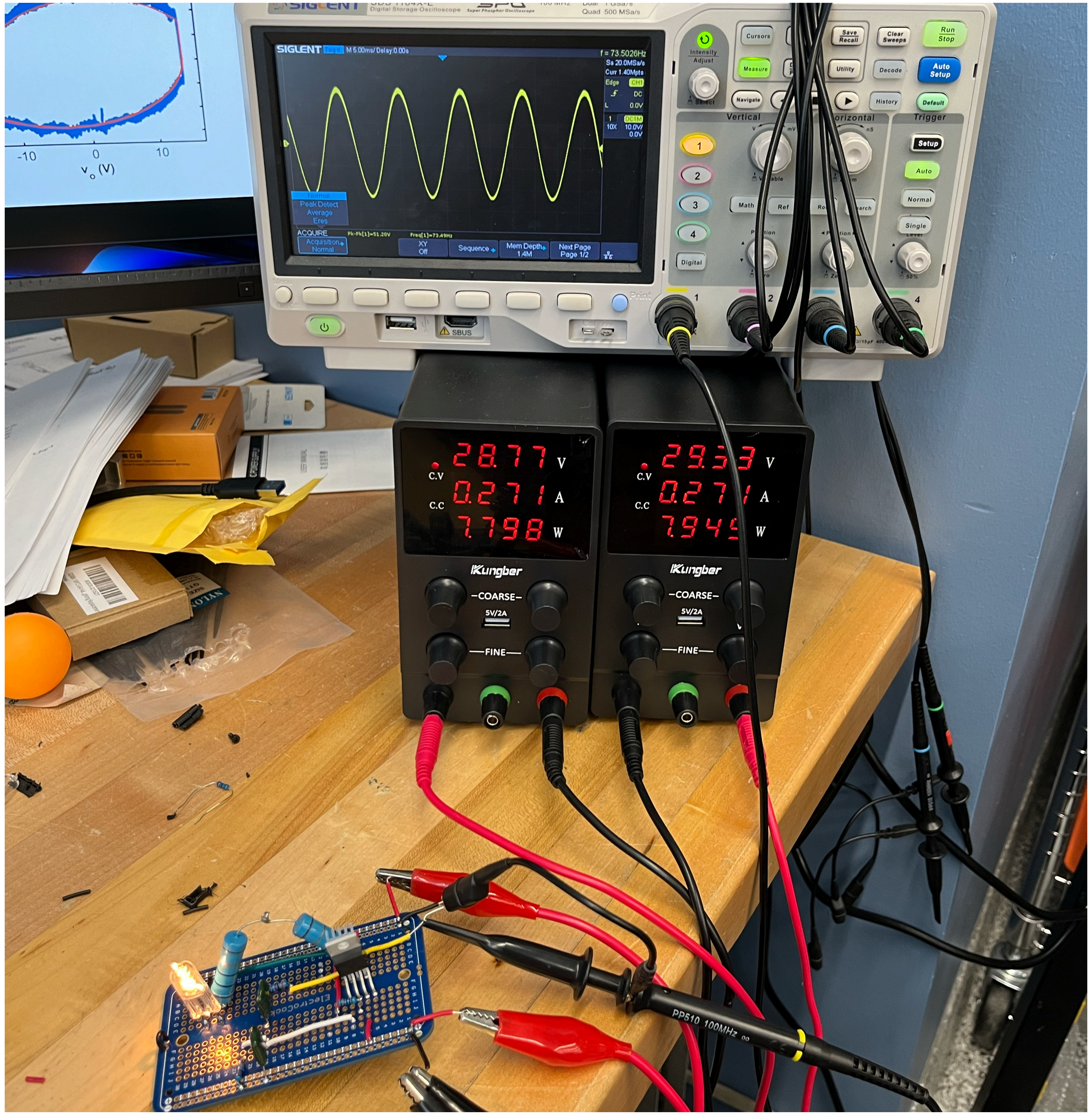
Nonlinear Analysis of the Wein Bridge Oscillator Circuit

18.385 Spring 2024



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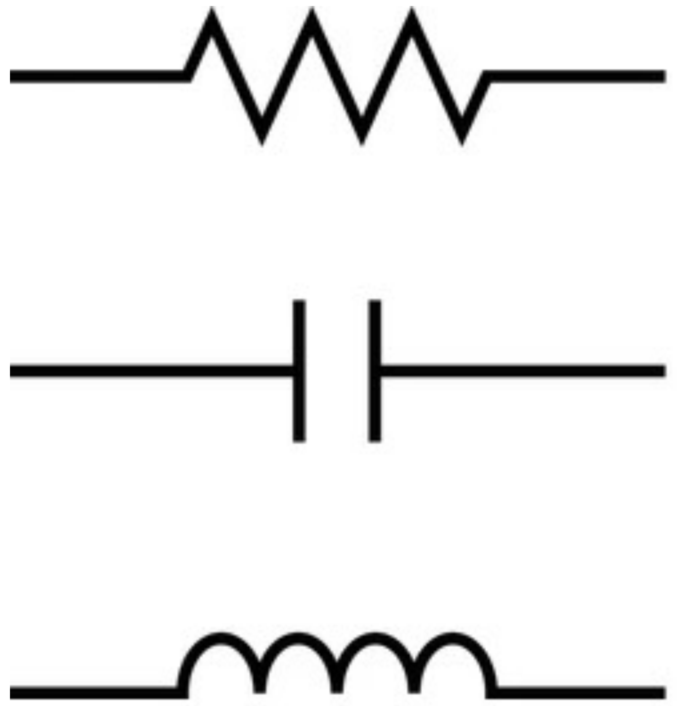
May 08, 2024

Electronic Circuits Primer

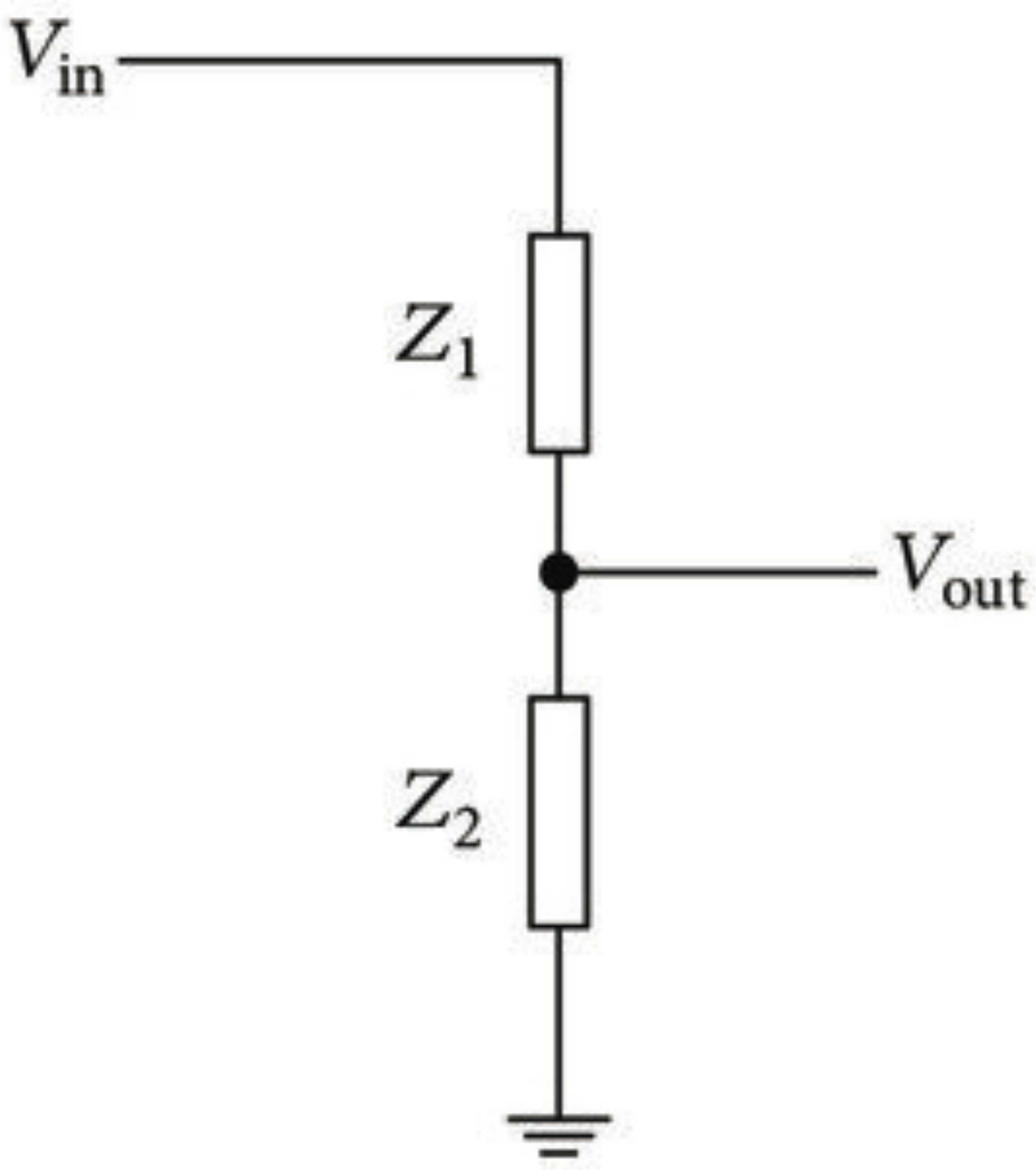


Resistor $$v = Ri$$

Capacitor $$\frac{dv}{dt} = \frac{1}{C} i$$

Inductor $$v = L \frac{di}{dt}$$

$$v = Ri \Leftrightarrow V(s) = RI(s) \implies Z_R = R$$
$$\frac{dv}{dt} = \frac{1}{C} i \Leftrightarrow V(s) = \frac{1}{sC} I(s) \implies Z_C = \frac{1}{sC}$$
$$v = L \frac{di}{dt} \Leftrightarrow V(s) = sLI(s) \implies Z_L = sL$$

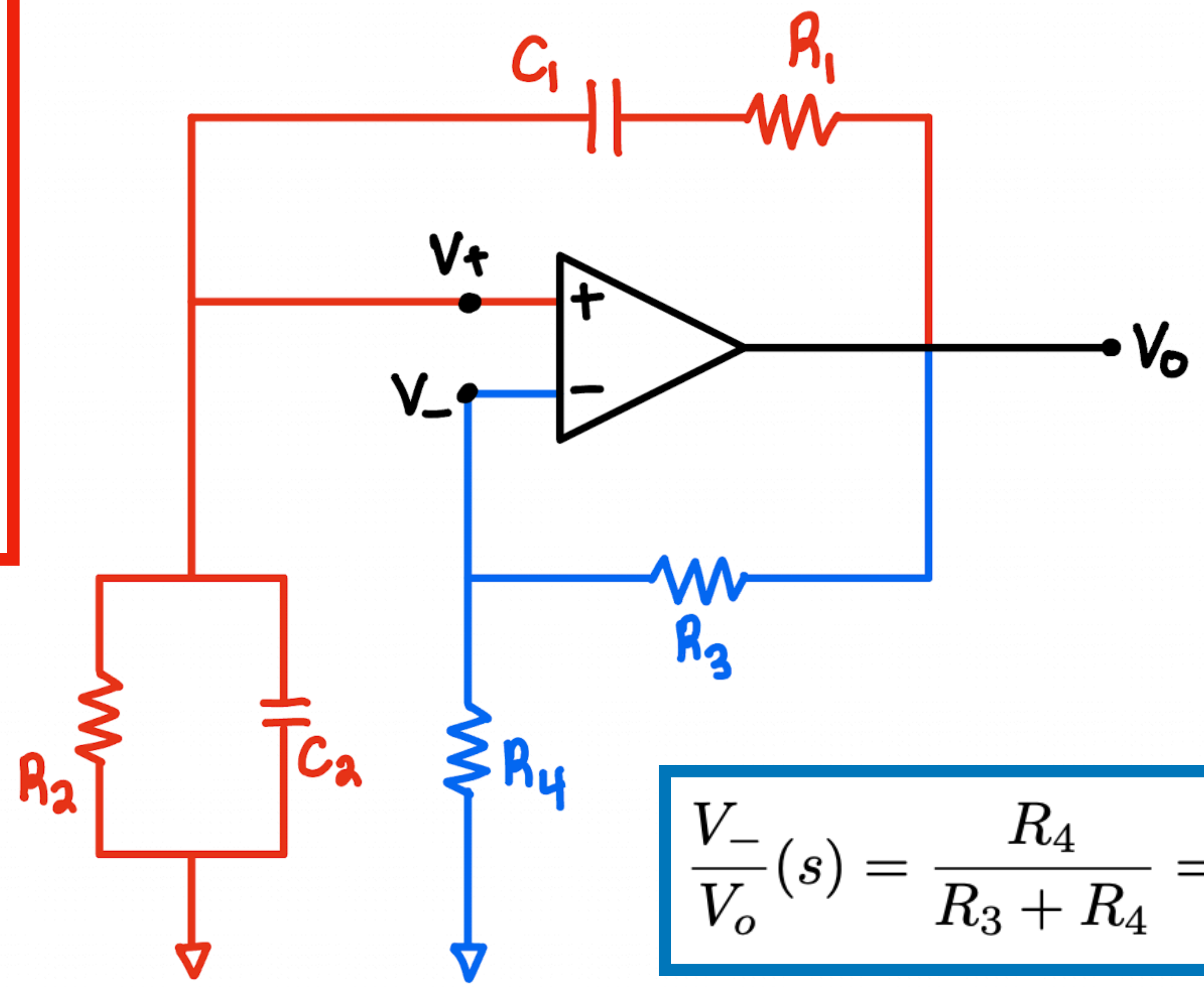


$$V_{source} = V_{in} - V_{out} = V_{in} \frac{Z_1}{Z_1 + Z_2}$$

$$V_{load} = V_{out} = V_{in} \frac{Z_2}{Z_1 + Z_2}$$

Wein Bridge Oscillator

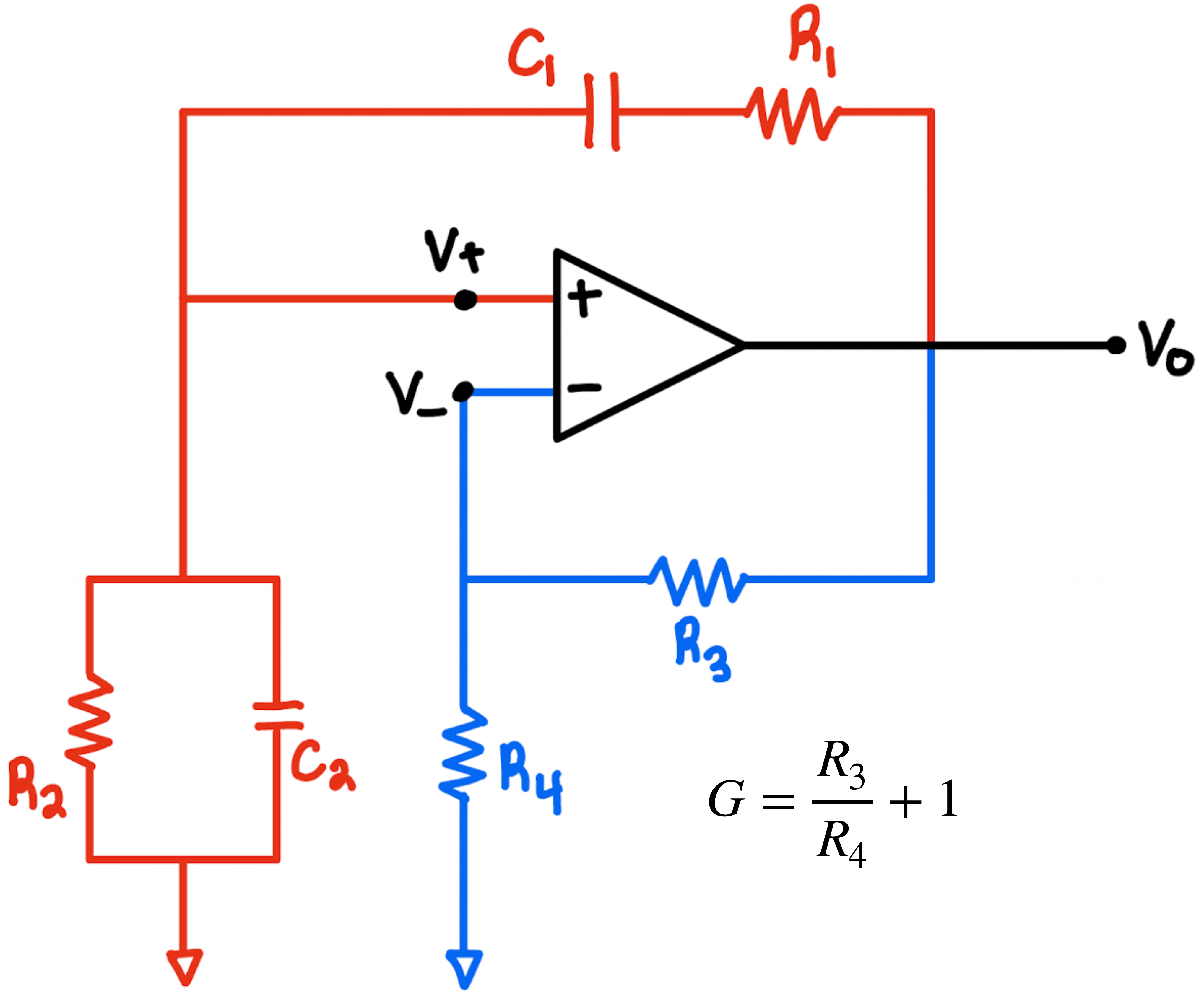
$$\frac{V_+}{V_o}(s) = \frac{\dfrac{1}{1/R_2 + 1/\frac{1}{sC_2}}}{R_1 + \frac{1}{sC_1} + \dfrac{1}{1/R_2 + 1/\frac{1}{sC_2}}}$$
$$= \frac{sR_2C_1}{s^2R_1C_1R_2C_2 + s(R_1C_1 + R_2C_2 + R_2C_1) + 1}$$
$$= \frac{s\tau_{21}}{s^2\tau_1\tau_2 + s(\tau_1 + \tau_2 + \tau_{21}) + 1} \quad (6)$$



$$v_o = A(v_+ - v_-) \Leftrightarrow V_o(s) = A(V_+(s) - V_-(s))$$

$$\frac{V_-}{V_o}(s) = \frac{R_4}{R_3 + R_4} = \frac{1}{G}$$

Linear Analysis



$$G = \frac{R_3}{R_4} + 1$$

$$V_o = A(V_+ - V_-)$$

$$0 = V_o \left(s^2 \tau_1 \tau_2 + s \left(\tau_1 + \tau_2 + \tau_{21} - \frac{GA}{G+A} \tau_{21} \right) + 1 \right)$$

$$= \tau_1 \tau_2 \ddot{v}_o + \left(\tau_1 + \tau_2 + \tau_{21} - \frac{GA}{G+A} \tau_{21} \right) \dot{v}_o + v_o. \quad (9)$$

$$0 = \frac{1}{\omega_n^2} \ddot{x} + \frac{2\zeta}{\omega_n} \dot{x} + x.$$

$$x(t) = A_0 e^{-\omega_n \zeta n} \sin \left(\omega_n \sqrt{\zeta^2 - 1} t + \phi_0 \right)$$

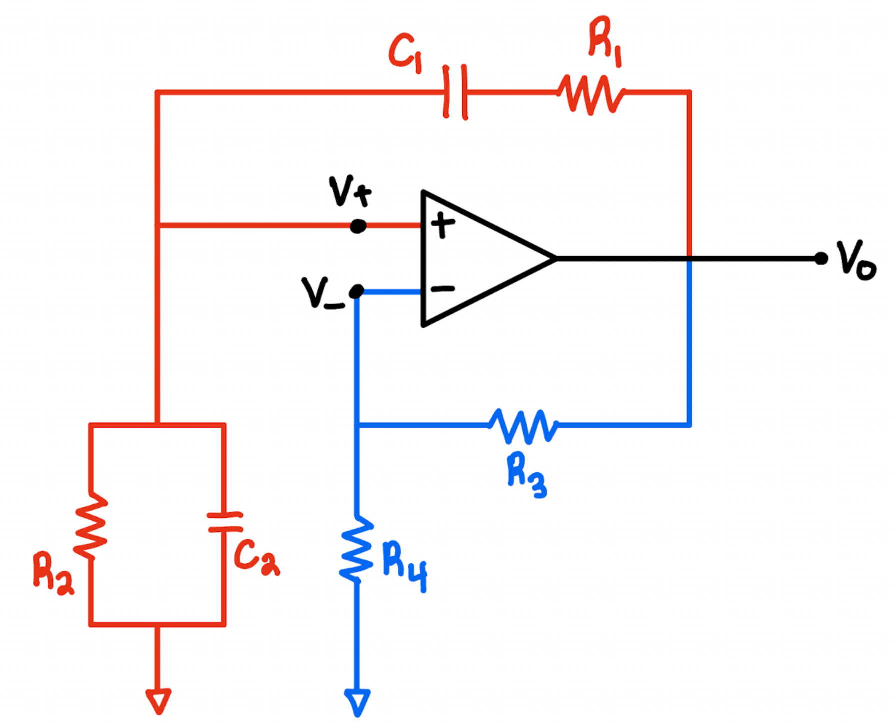
<u>for constant-amplitude oscillations</u>

$$0 = \tau_1 + \tau_2 + \tau_{21} - \frac{G_* A}{G_* + A} \tau_{21}$$

$$= \tau_1 + \tau_2 + \tau_{21} - G_* \tau_{21} \text{ for } A \gg G_*$$

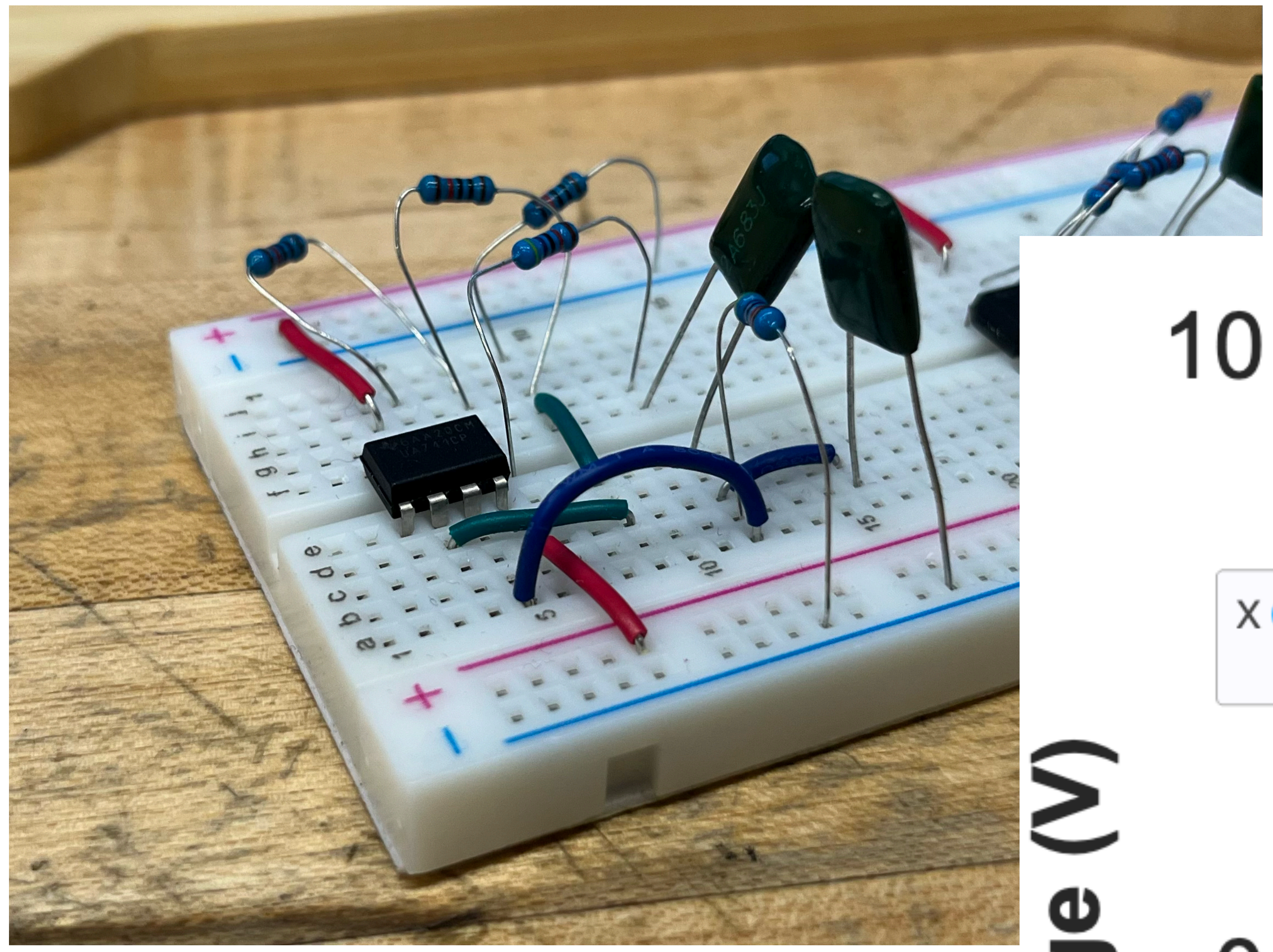
$$\implies G_* = \frac{\tau_1 + \tau_2}{\tau_{21}} + 1$$

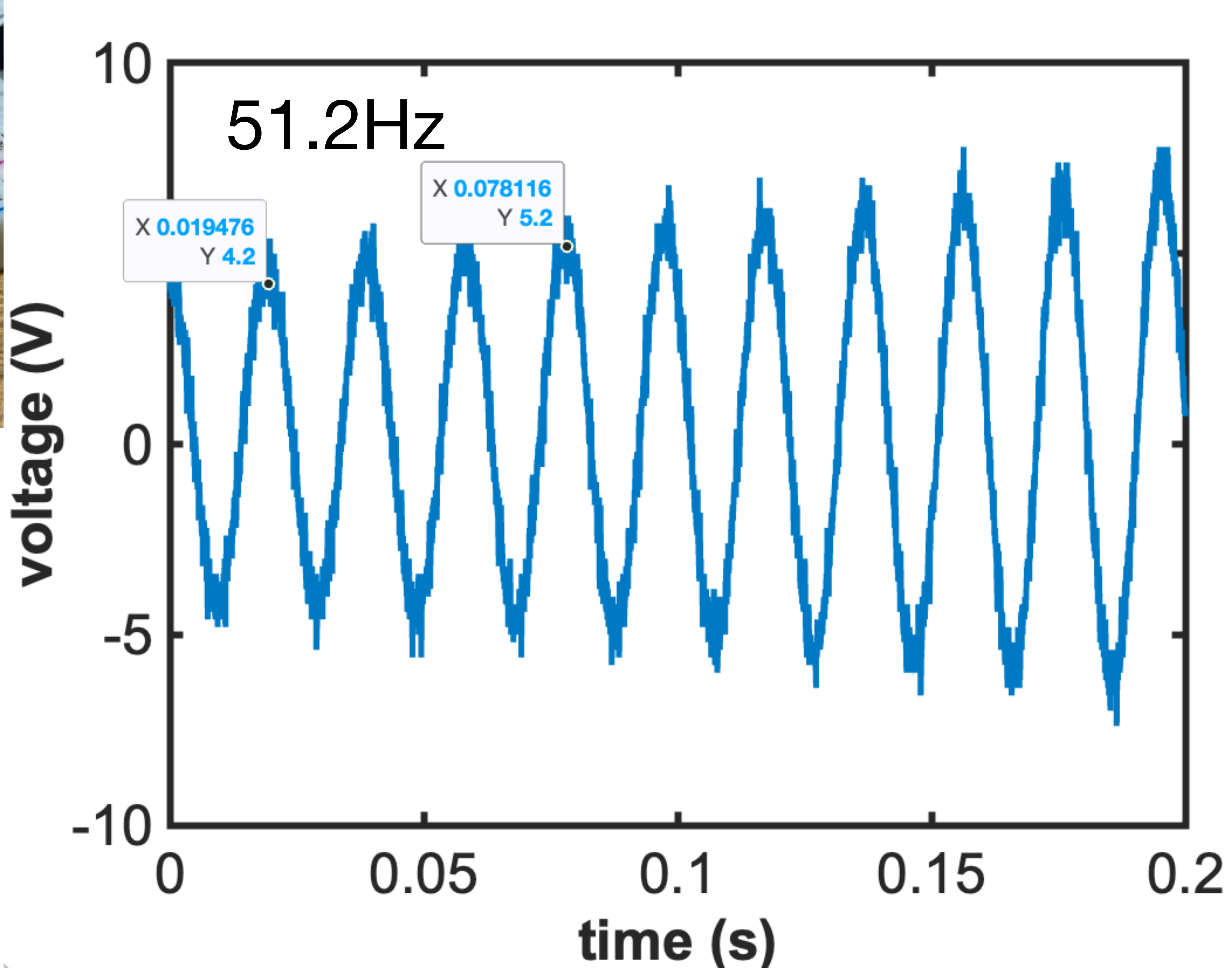
Hardware Experiment



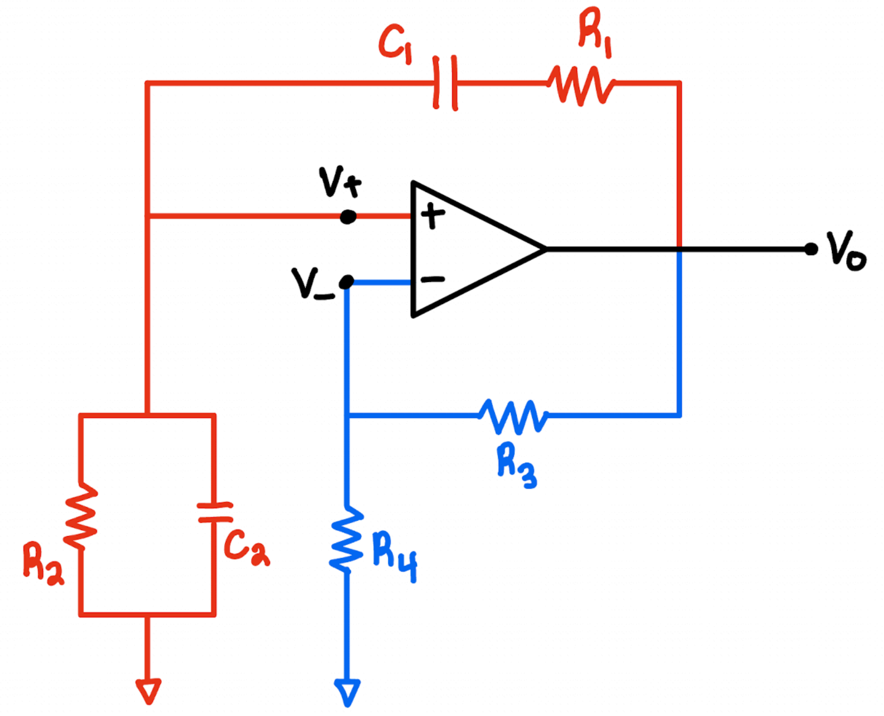
component	value
R_1	47 kΩ
C_1	68 nF
R_2	47 kΩ
C_2	68 nF
R_3	20 kΩ
R_4	10 kΩ

$\zeta = 0; \omega_n = \omega_d = 49.8\text{Hz}$

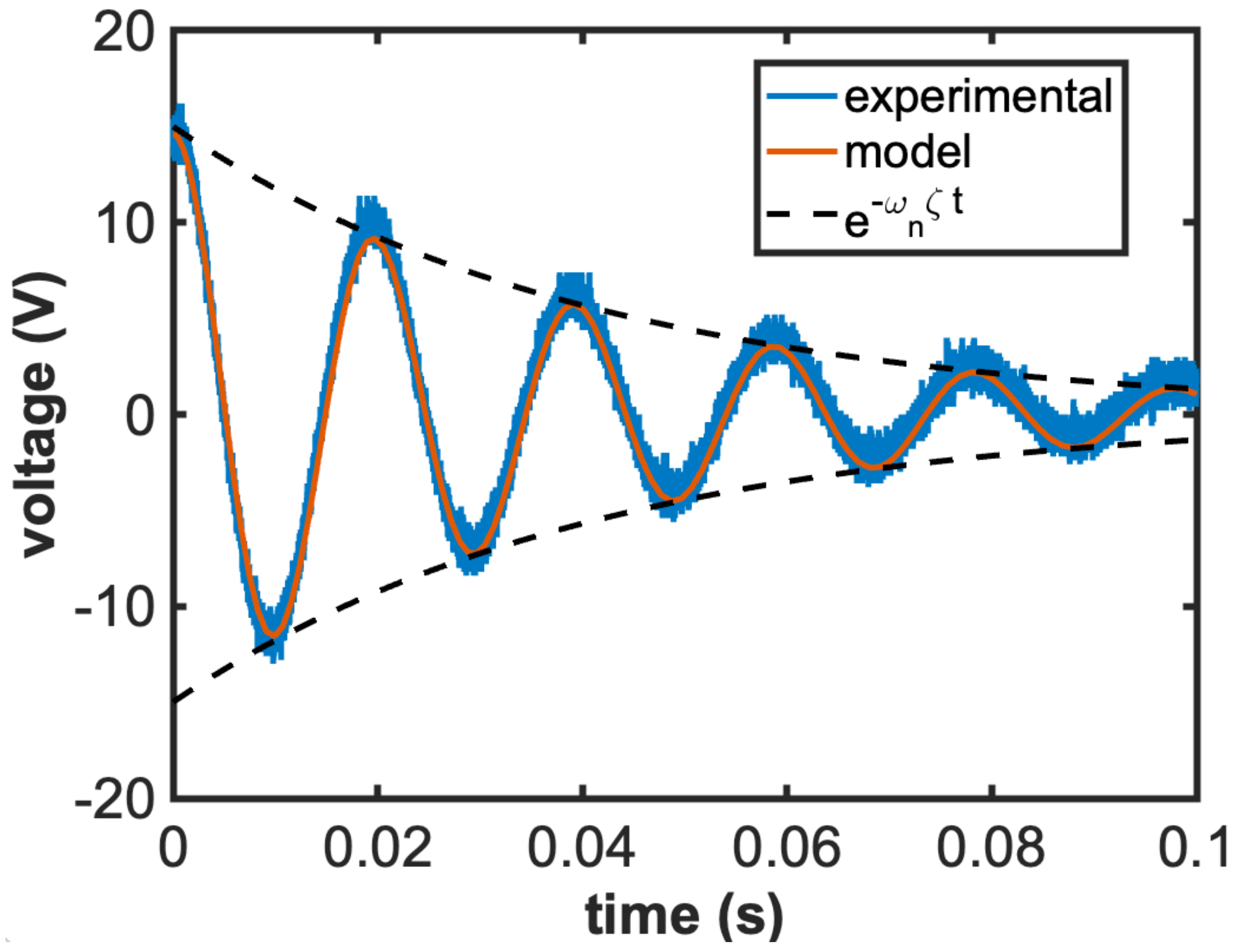




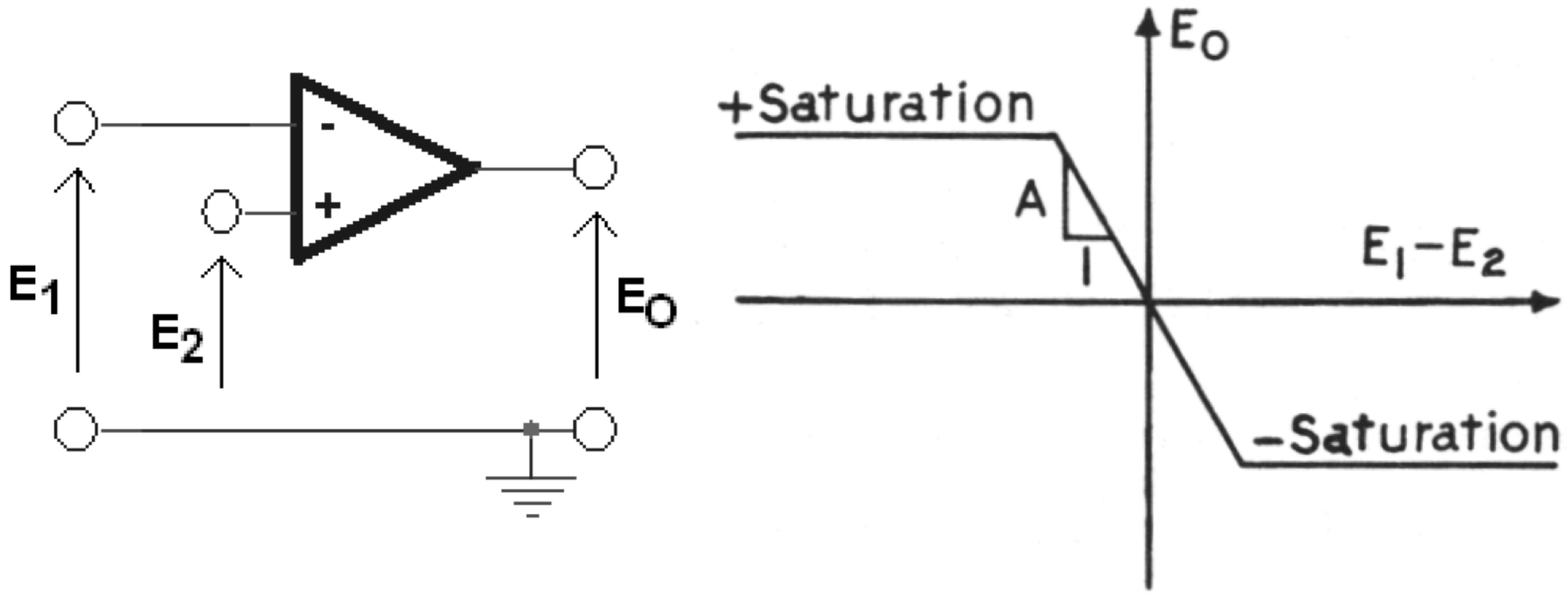
Component Tolerances



component	nominal value	
R_1	47 kΩ	
C_1	68 nF	
R_2	47 kΩ	
C_2	68 nF	
R_3	15kΩ	14.77kΩ
R_4	8kΩ	8.082kΩ



Unintentional Nonlinearity: Saturation



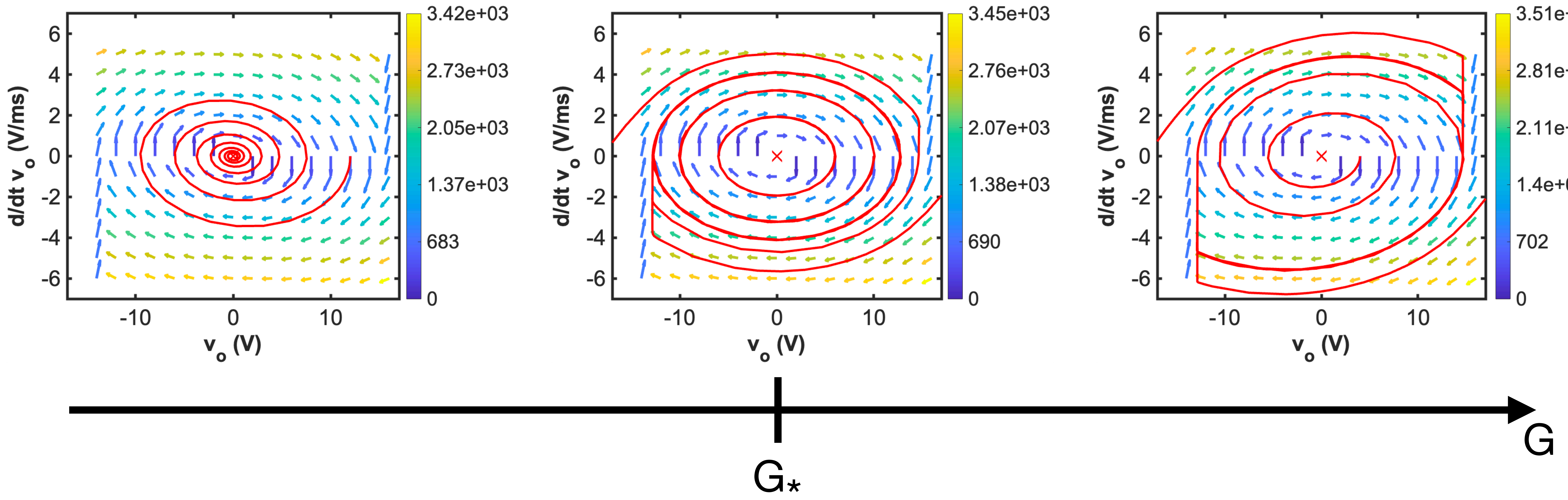
$$s(x) = x\left(1 - \sigma(-(x - X_l))\sigma(-x) - \sigma(x - X_u)\sigma(x)\right) + 500\left((X_l - x)\sigma(-(x - X_l)) + (X_u - x)\sigma(x - X_u)\right)$$

$$\begin{aligned} \sigma(x) &= \text{smooth step function } \tfrac{1}{1+e^{-100x}} \\ X_l &= \text{lower saturation bound} \\ X_u &= \text{upper saturation bound} \end{aligned}$$

$$\begin{cases} \dot{v}_o = \dot{v}_o \\ \ddot{v}_o = -\dfrac{(\tau_1 + \tau_2 + \tau_{21} - G\tau_{21})\dot{v}_o + v_o}{\tau_1 \tau_2} \end{cases} \Rightarrow \begin{cases} \dot{v}_o = s(\dot{v}_o) \\ \ddot{v}_o = -\dfrac{(\tau_1 + \tau_2 + \tau_{21} - G\tau_{21})\dot{v}_o + v_o}{\tau_1 \tau_2} \end{cases}$$

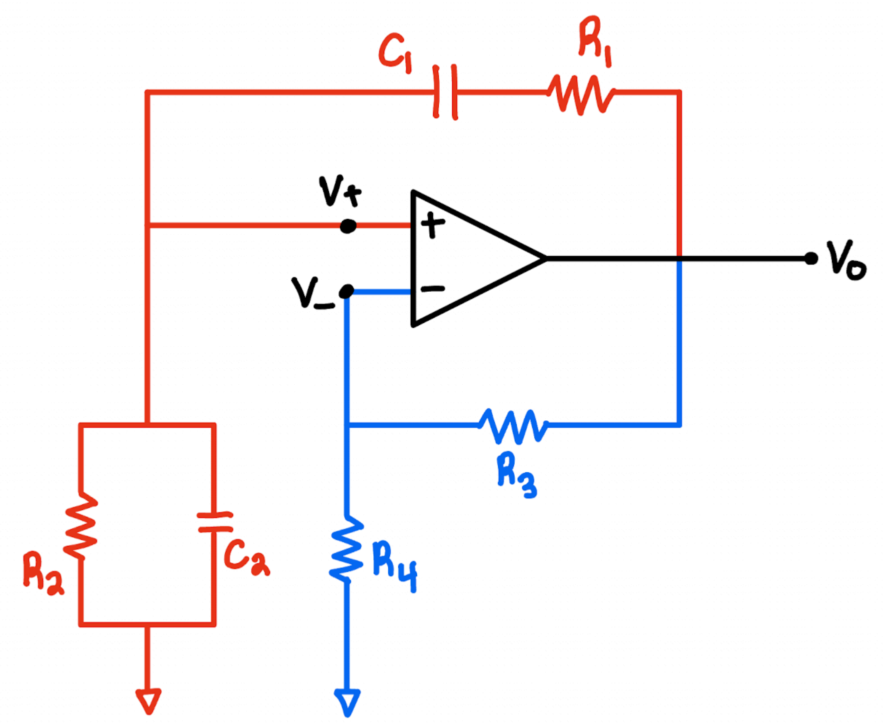
G as a Parameter

$$G_* = \frac{\tau_1 + \tau_2}{\tau_{21}} + 1$$

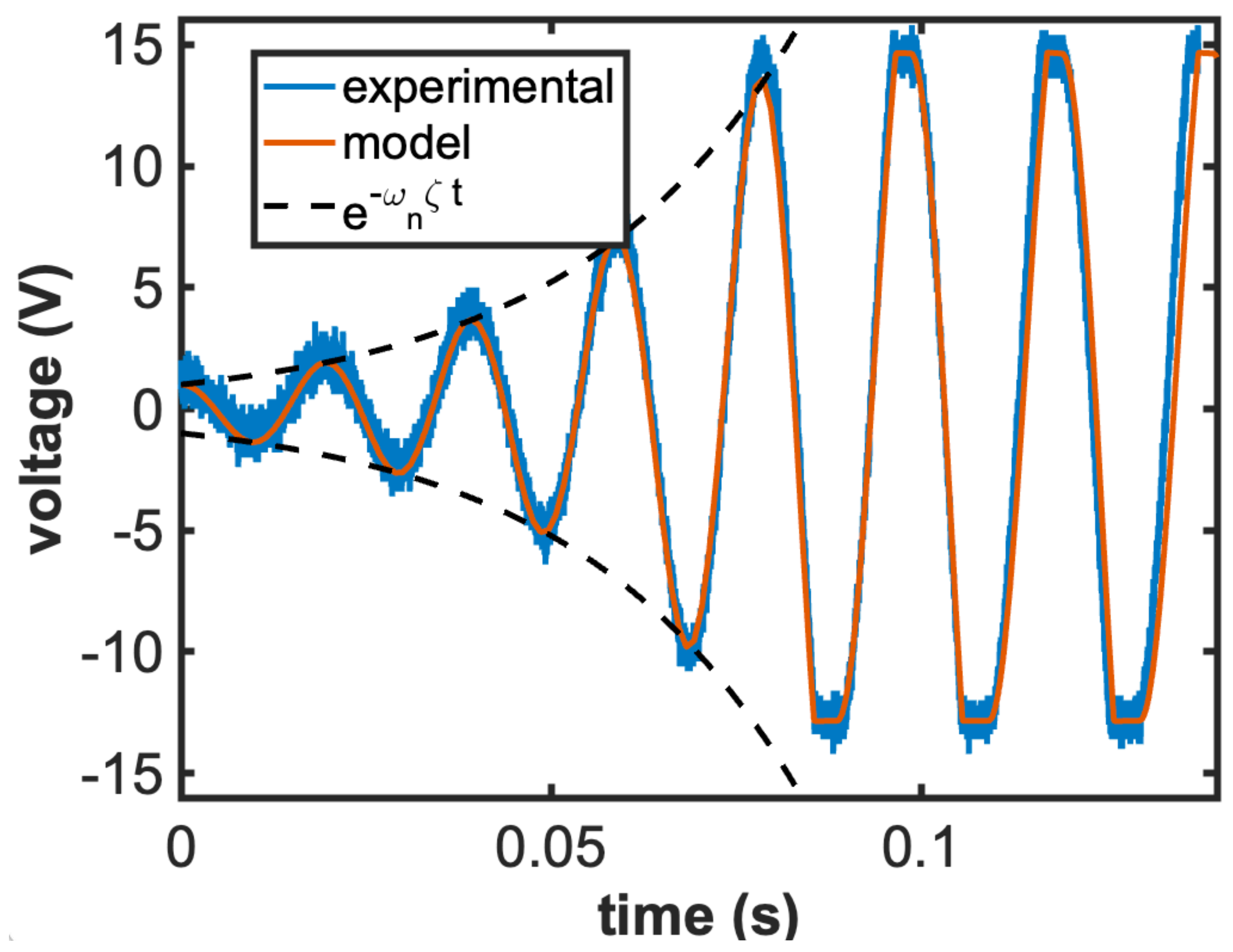


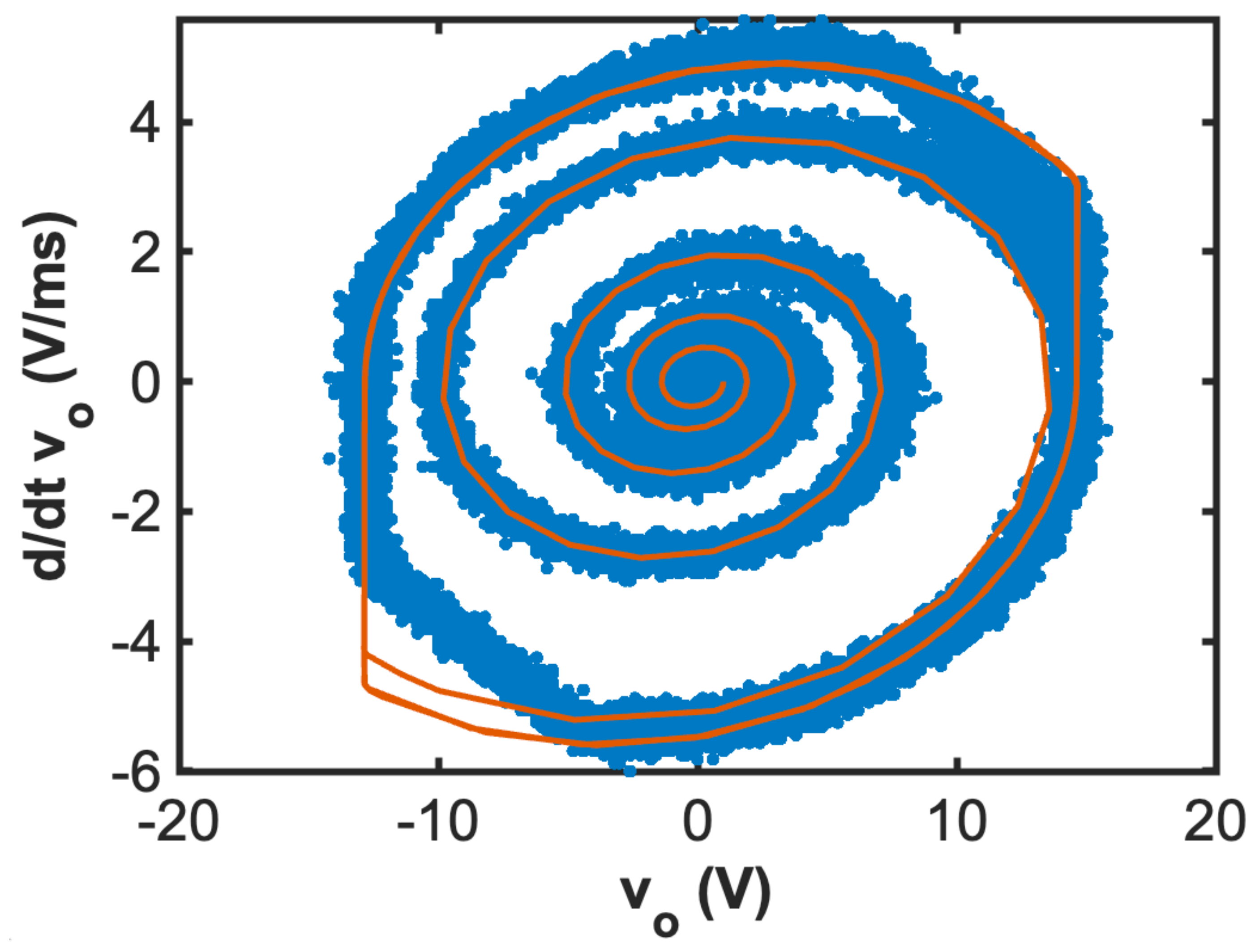
degenerate Hopf Bifurcation

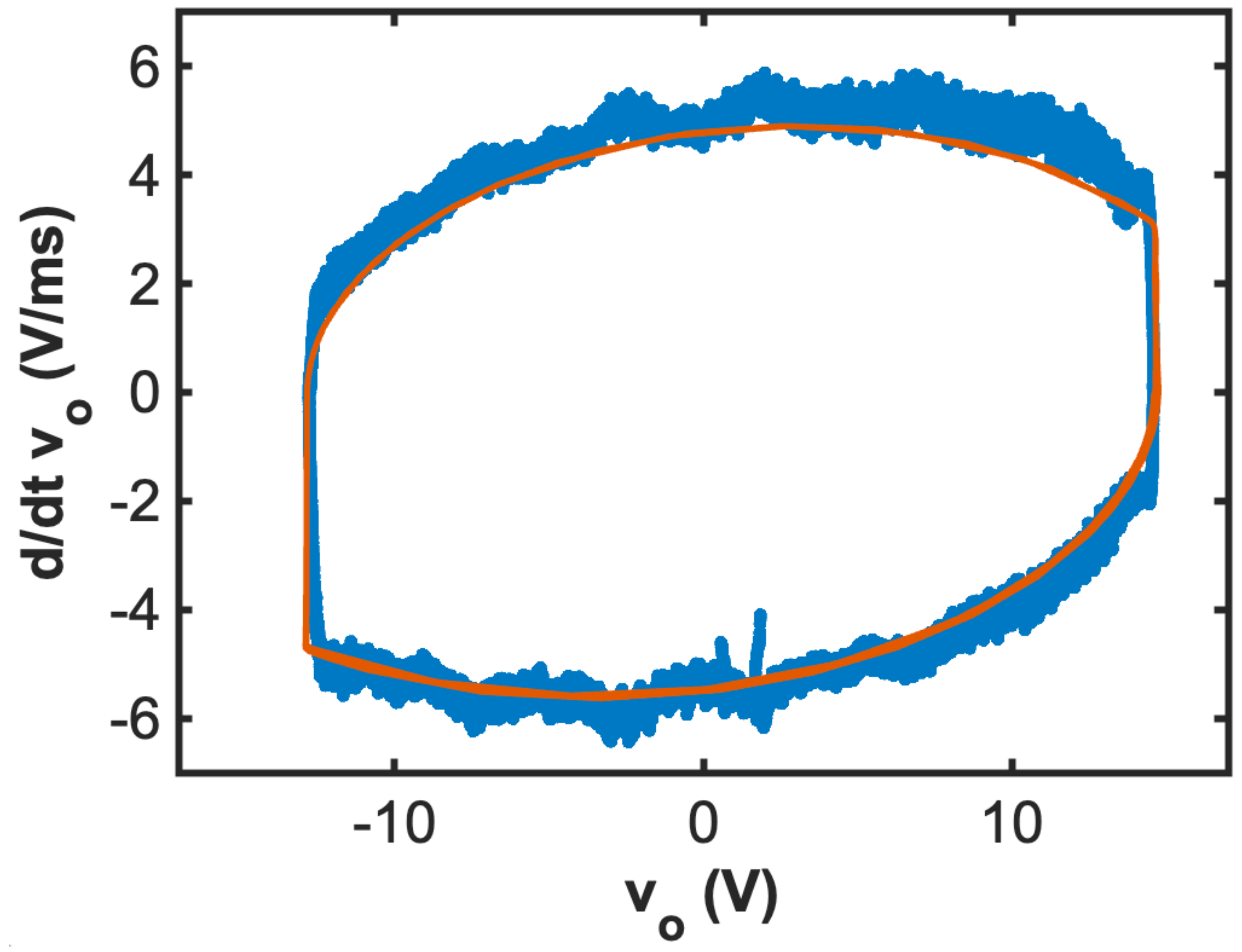
Hardware Experiment



component	nominal value	measured value
R_1	47 kΩ	46.54 kΩ
C_1	68 nF	68.12 nF
R_2	47 kΩ	46.14 kΩ
C_2	68 nF	65.93 nF
R_3	22 kΩ	21.52 kΩ
R_4	10 kΩ	9.875 kΩ







Nonlinear Feedback

$$0 = \tau_1\tau_2\ddot{v}_o + (\tau_1 + \tau_2 + \tau_{21}(G - 1))\dot{v}_o + v_o$$

function of oscillation amplitude?

$$G = \frac{R_3}{R_4} + 1$$

increase when amplitude is too high
decrease when amplitude is too low

more voltage → more heat

more heat → more resistance

more resistance → lower gain

(and vice versa)

Lightbulb Constitutive Equations

$$R(T) = R_0(1 + \alpha(T - T_0)) \tag{14}$$

where:

T = temperature (K)
α = temperature coefficient of resistance (Ω/K)
T_0 = room temperature (measured to be 295K)
R_0 = resistance (Ω) measured at room temperature

$$mc\dot{T} = \frac{v_R^2}{R(T)} - Q(T). \tag{15}$$

where:

m = mass of filament (kg)
c = specific heat of filament (J/kgK)
v_R = voltage (V) across resistor
Q = heat emitted by bulb (W)

$$Q(T) = \epsilon\sigma A_s(T^4 - T_0^4). \tag{16}$$

where:

ϵ = thermal emissivity coefficient
σ = Steffan-Boltzmann constant (W/m$^2K^4$)
A_s = surface area of filament (m^2)

Replacing R_4 with $R(T)$

$$\begin{cases} \dot{v}_o = \dot{v}_o \\ \ddot{v}_o = -\dfrac{(\tau_1 + \tau_2 + \tau_{21} - G(T)\tau_{21})\dot{v}_o + v_o}{\tau_1 \tau_2} \\ \dot{T} = \dfrac{1}{mc}\left(\dfrac{R(T)}{(R_3 + R(T))^2} v_o^2 - Q(T)\right). \end{cases}$$

$$v_o = Ae^{-\omega_n \zeta t} \sin(\omega_d t + \phi(t))$$

$$\langle v_o \rangle = \frac{1}{\sqrt{2}} A$$

$$G(T) = 1 + \frac{R_3}{R(T)}$$

↓ reduced model ↓

$$\begin{cases} \dot{A} = -\zeta(T)\omega_n A \\ \dot{T} = \dfrac{1}{mc}\left(\dfrac{R(T)}{(R_3 + R(T))^2} \dfrac{A^2}{2} - Q(T)\right) \end{cases}$$

$$\omega_n = \frac{1}{\sqrt{\tau_1 \tau_2}}; \; \zeta(T) = \frac{\omega_n}{2}(\tau_1 + \tau_2 + (1 - G(T))\tau_{21}),$$

Steady-state Coefficient Fitting

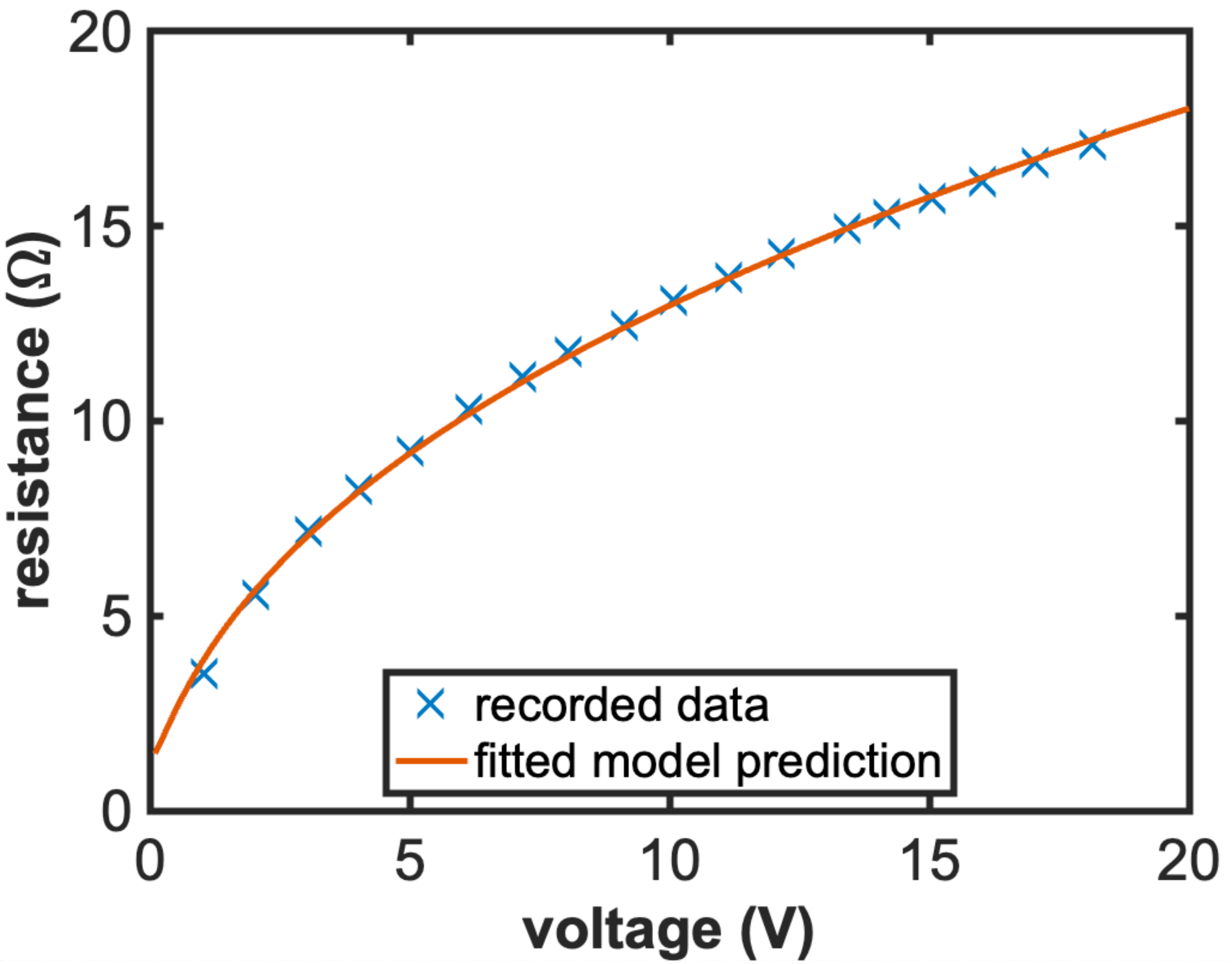
$$mc\dot{T} = \frac{v_R^2}{R(T)} - Q(T)$$

$$Q(T) = \epsilon\sigma A_s(T^4 - T_0^4)$$

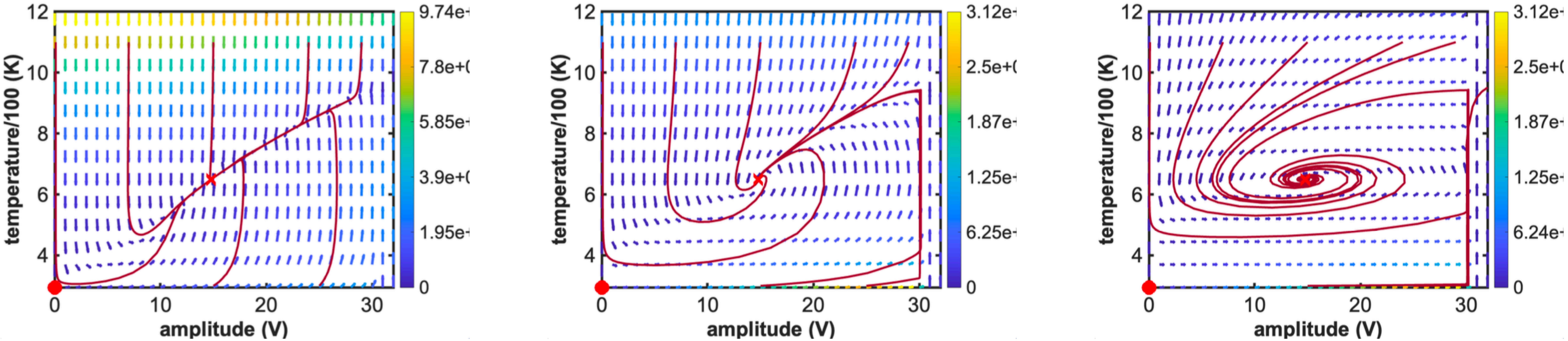
$$R(T) = R_0(1 + \alpha(T - T_0))$$

$$\dot{T} = 0 \implies v_R = \sqrt{R(T)Q(T)}$$

$$R_0 = 1.35\Omega$$
$$\alpha = .0132$$
$$\epsilon A_s = 1.692 \cdot 10^{-4}$$



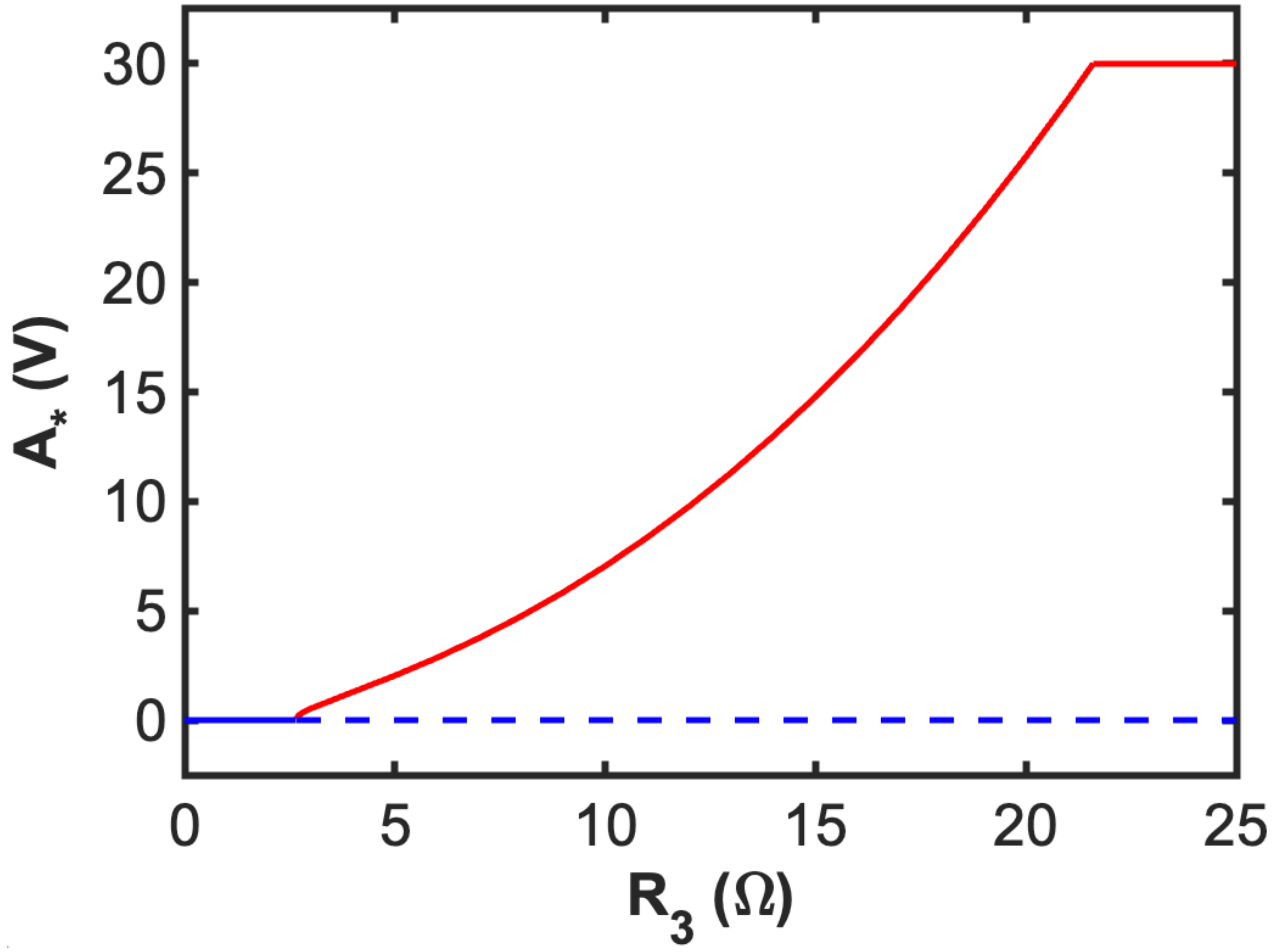
Effect of mc

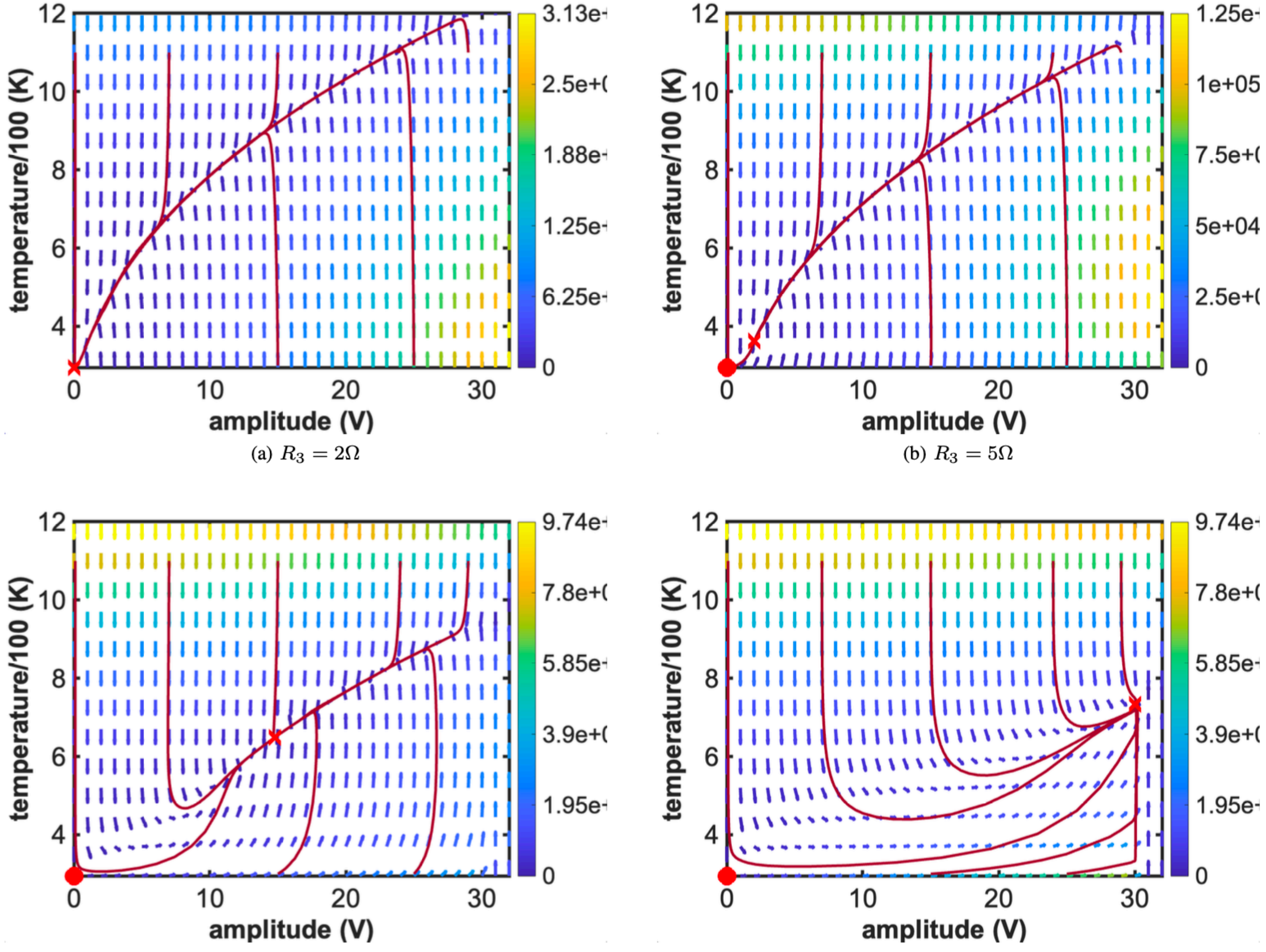


(a) $mc = 10^{-6}$ (b) $mc = 10^{-5}$ (c) $mc = 10^{-4}$

R_4 as a Parameter

$$G(T) = 1 + \frac{R_3}{R(T)}$$





(a) $R_3 = 2\Omega$

(b) $R_3 = 5\Omega$

(c) $R_3 = 15\Omega$

(d) $R_3 = 30\Omega$

Experimental Validation